Research statement

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I'm exploring applications of the internal language of toposes in algebraic geometry and commutative algebra. These uncover certain specific relations between logic, algebra, and geometry, and open up new and unique perspectives.

Toposes are special kinds of categories, an important example being the category $\mathrm{Sh}(X)$ of set-valued sheaves on a topological space X. Their internal language allows to speak and reason about the objects and morphisms of a topos in a naive element-based language: From the internal perspective, objects of the topos look like sets, morphisms look like maps between sets, epimorphisms look like surjective maps, monomorphisms look like injective maps, group objects look like plain groups, and so on; and any theorem which has a constructive proof also holds in the internal universe of a topos.

A basic example is as follows. One can check that a sheaf of modules on a scheme X is of finite type if and only if, from the internal point of view of the topos $\mathrm{Sh}(X)$, it is a finitely generated module. The standard proof of the theorem "if the two outer modules in a short exact sequence of modules are finitely generated, then so is the middle one" is constructive. Therefore the theorem holds in any topos. Interpreted in the topos $\mathrm{Sh}(X)$ it yields the theorem "if the two outer sheaves of modules in a short exact sequence of sheaves of modules are of finite type, then so is the middle one".

In this way, the internal language of toposes gives a precise connection between commutative algebra and algebraic geometry: (Some) concepts and statements of algebraic geometry are simply interpretations of concepts and statements of commutative algebra in the internal language of a suitable topos. This observation allows to skip over routine proofs, brings conceptual clarity, and rigorously justifies certain kinds of "fast and loose reasoning" – verifying a theorem only in the affine case without properly working out the general case or constructing a sheaf only over affine open subsets without meticulously verifying the gluing condition.

I believe that these kinds of applications are already useful to working algebraic geometers. However, more advanced applications are also possible. They result from considering internal statements whose logical form is more complex and whose external meaning is therefore not obvious, and from internal statements whose proofs exploit unique features of the internal universes.

For instance, if X is a reduced scheme, the internal universe of $\mathrm{Sh}(X)$ has the peculiar feature that the structure sheaf $\mathcal{O}_X$ is Noetherian and a field, even if X is not locally Noetherian and (as will almost always be the case) the local rings $\mathcal{O}_{X,x}$ are not fields. This fact has no simple external counterpart; it's rather an intricate statement about the interplay between the rings $\Gamma(U, \mathcal{O}_X)$ for varying open subsets $U \subseteq X$.

Thanks to this particular feature, linear and commutative algebra over $\mathcal{O}_X$ are particularly simple from the internal point of view. For instance, Grothendieck's generic freeness lemma, which is usually proved using a somewhat involved series of reduction steps, admits a short, easy, and conceptual proof with this technique [2, Section 11.5], since the freeness lemma is trivial for fields. This new proof is even constructive, which is quite surprising, since the previously known proofs didn't suggest at all that a constructive proof would be possible.

It is in this way that the internal language unlocks new approaches: by making concepts accessible which would otherwise be too unwieldy to manage and by allowing to import a huge corpus of prior work, namely the entire literature on constructive algebra.

Building a dictionary

There was a flurry of activity on the internal language machinery in the 1970s, when it was worked out and applications were discovered; a very accessible introduction to the internal language of that time is written by Mulvey [9], culminating in an internal proof of the Serre–Swan theorem. However, for the internal language to be truly useful in algebraic geometry, one needs to have an extensive dictionary relating internal and external notions. In the 1970s, only few such dictionary entries were known; a notable exception is an internal characterization of the étale topos by Wraith [17].

A major task of my PhD studies was therefore to systematically search for internal translations of well-known external concepts. Some were easy to obtain; others required creative work, were surprising, and depended on certain subtleties. An excerpt of the now-known dictionary pertaining to a scheme X is as follows:

externally	internally to $\mathrm{Sh}(X)$
sheaf of sets	set
morphism of sheaves	map of sets
monomorphism	injective map
epimorphism	surjective map
sheaf of rings	ring
sheaf of modules	module
sheaf of finite type	finitely generated module
finite locally free sheaf	finite free module
coherent sheaf	coherent module
quasicoherent sheaf	module satisfying certain sheaf conditions
injective sheaf	injective module
tensor product of sheaves	tensor product of modules
rank function of a sheaf of modules	minimal number of generators
dimension of X	Krull dimension of $\mathcal{O}_X$
sheaf of rational functions	total quotient ring of $\mathcal{O}_X$
relative spectrum of a sheaf of algebras	variant of the ordinary absolute spectrum
big Zariski topos of X	topos classifying local $\mathcal{O}_X$-algebras which are local over $\mathcal{O}_X$
fppf topos of X	topos classifying fppf-local $\mathcal{O}_X$-algebras which are local over $\mathcal{O}_X$

Finding the internal translation of the quasicoherence condition turned out to be quite important for the theory, because it paved the way for the discovery of the internal translation of the relative spectrum construction (the first step in reinterpreting relative algebraic geometry over a base scheme as absolute algebraic geometry over a point) and because it explained the proper background of the fact that $\mathcal{O}_X$ looks like a field from the internal point of view (if X is reduced). The field property was already observed in the 1970s, by Mulvey; however, its full potential wasn't realized. Tierney commented at that time [15, page 209]:

“[It] is surely important, though its precise significance is still somewhat obscure”. We now know that the field property is an immediate consequence of the deeper fact that $\mathcal{O}_X$ satisfies the internal translation of being quasicoherent.

I also found a general metatheorem relating properties of an A-module M with the internal properties of the induced quasicoherent sheaf $M^\sim$ on $\mathrm{Spec}(A)$ and I found a way of using the internal language of a topos to speak about its subtoposes; the latter is useful for studying spreading of properties from points to open neighborhoods.

In the future, I want to extend this dictionary, in particular to terms of intersection theory and of the theory of derived categories of coherent sheaves; to study the properties of the internal universe in case not the Zariski topology, but finer topologies such as the étale, fppf, or ph topologies are employed; to determine explicit descriptions of the geometric theories which the various big toposes of a scheme classify (for the Zariski and the étale topology, these are well-known; for the fppf and the surjective topology, I obtained descriptions in my PhD studies, subsuming some aspects of [13] and [5]; and all other cases are unknown); to find applications in algebraic geometry and commutative algebra; and to further develop a constructive account of algebraic geometry.

Synthetic algebraic geometry

The internal language machinery also allows to develop a synthetic account of algebraic geometry, similar to the existing synthetic accounts of differential geometry [7], domain theory [6], computability theory [1], and more recently and very successfully homotopy theory [16] and related subjects [11, 12, 10]. The synthetic approaches allow in each case to encode the objects of study directly as (nonclassical) sets, with geometric, domain-theoretic, computability-theoretic, or homotopy-theoretic structure being automatically provided for.

The home for synthetic algebraic geometry over a base scheme S is the internal universe of the big Zariski topos of S. In the toposes used for synthetic differential geometry, the statement “any set-theoretic map $\mathbb{R} \to \mathbb{R}$ is smooth” is true, appropriately formulated. In the big Zariski topos, the statement “any set-theoretic map $\mathbb{A}^1_S \to \mathbb{A}^1_S$ is a polynomial function” is true. This property of the affine line over S endows the internal universe with a distinctive algebraic flavor and neatly captures the general intuition we have about algebraic geometry: “Everything is polynomial.”

In my PhD studies, I found internal definitions of the concepts of affine schemes and general schemes, open and closed immersions, quasicompact and quasiseparated morphisms, universally closed and proper morphisms, and several related notions. Central to the theory is “synthetic quasicoherence”, an internal rendition of what it means for a sheaf of modules to be quasicoherent; all other notions depend on this, and all known internal properties of $\mathbb{A}^1_S$ (such as its field property – first discovered by Kock [8] – or that it is algebraically closed in a weak sense) follow from the fact that $\mathbb{A}^1_S$ is synthetically quasicoherent.

The synthetic account is not nearly as well-developed as the synthetic account of differential geometry, its closest cousin. In the future, I want to further the theory with the goal of interpreting a nontrivial amount of algebraic geometry in the synthetic setting; to find additional interesting properties of $\mathbb{A}^1_S$; to understand whether it’s indeed the case that all properties of $\mathbb{A}^1_S$ follow from its synthetic quasicoherence and if so, in which precise sense; to internally characterize subtoposes of the big Zariski topos, corresponding to finer topologies; and to extract applications in algebraic geometry, particularly pertaining to topologies other than the Zariski topology.

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